PERGAMON

International Journal of Heat and Mass Transfer 43 (2000) 3965–3975

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Analytical method in inverse heat transfer problem using Laplace transform technique

Masanori Monde

Department of Mechanical Engineering, Saga University, 1 Honjo-machi, Saga-shi, Saga 840-8502, Japan

Received 14 June 1999; received in revised form 20 January 2000

Abstract

An analytical method has been developed for the inverse heat conduction problem, when the temperatures are known at two positions in a finite body or at one position in a semi-infinite body. On the basis of these known temperatures, a closed form solution is determined for the transient temperatures beyond the two positions by using Laplace transform technique. This method first approximates the temperature data with a half polynomial power series of time. The resultant expression for an objective temperature or heat flux is explicitly obtained in the form of power series of time. Numerical results for some representative problems show that the surface temperature and heat flux can be predicted well by the method.

1. Introduction

A procedure to solve inverse heat transfer problem (IHTP) is very important in determining unknown surface temperature and heat flux from known values in the body, which are usually measured as a function of space and time. Especially, under severe surface conditions such as re-entry of space vehicle and accidents involving coolant breaks in the plasma-facing components, a direct measurement of the heat flux or surface temperature change on the surface is almost impossible, so that the prediction of these values cannot help, depending on the solution of IHTP. In addition, several studies about IHTP have been carried out to predict the transient surface conditions during quenching of a hot body. However, the exact solution for IHTP is mathematically verified that does not exist within a certain time depending on the position at which the known value is provided. Therefore, recent studies of IHTP have been numerically treated and extended to multiple dimensions with the help of computing architechture and improvement in computer capacities. Several numerical and theoretical approaches to IHTP are summarized elsewhere [1,2].

Nevertheless, a theoretical method, for example, using Laplace transformation, is still interesting because it not only explicitly gives the inverse solution but also greatly reduces the computing time. The disadvantage of this approach may be limited only to the cases where the configurations involved are rather simple such as rectangular and cylindrical shapes, and the known boundary conditions are not complicated. As an example of analytical method of one-dimensional IHTP, a procedure using an exact solution by Buggraf [3] and a method using Duhamel solution or Laplace [4–7] transformation are widely used.

Sparrow et al. [4] succeeded in deriving the inverse solution for one-dimensional heat conduction by using a skillfully introduced arbitrary function. The values of this function are discretely given with

E-mail address: monde@me.saga-u.ac.jp (M. Monde).

0017-9310/00/$ - see front matter
PII: S0017-9310(00)00040-5

Nomenclature

a	thermal diffusivity	T	temperature
$f_1(\tau), f_2(\tau)$	functions of non-dimensional temperature at points ξ_1 and ξ_2, respectively	x	x-coordinate
k	rate of temperature rise	$\theta(\xi, \tau)$	non-dimensional temperature
L	characteristic length	λ	heat conductivity
$\min(\theta)$	minimum of significant number or a minimum division of measuring equipment	$\Phi(\xi, \tau)$	non-dimensional heat flux
N	degree of approximate polynomial	$\bar{\theta}(\xi, s)$	subsidiary value of θ
N_{sf}	order of significant number	$\bar{\Phi}(\xi, s)$	subsidiary value of Φ
q	heat flux	ξ	non-dimensional distance ($= X/L$, $\xi_1 < \xi_2$)
s	Laplace operator ($= p^2$)	τ	non-dimensional time ($= at/L^2$)
t	time	τ_1	minimum predictive time
		τ_i^*	non-dimensional time lag ($\mathrm{erfc}(\xi_i/2\sqrt{\tau_i^*}) = \min(\theta)$)

time, since these values are calculated from the known values measured, which are usually given as a function of time. During the calculation of these values, a special care is needed to prevent the oscillations at successive integrals of time. Non-dimensional minimum interval of time, $\Delta\tau$, was recommended to be 0.01. In addition to the existence of the minimum interval, good agreement between the given and predicted surface temperatures for a linear increase in the surface temperature was obtained, although what level of accuracy was used for known values is not described.

Shoji [5] started from the same inverse solution in a subsidiary form as Sparrow et al. [4] and obtained two different types of solutions as a function of time in two different ways. Shoji [5] verified that one of the two solutions corresponds to the solution proposed by Buggraf [3], and the other to one obtained by Sparrow. Shoji [5] employed the data including uncertainties in the known values to evaluate the predictive accuracy of the inverse solution. He numerically calculated the inverse solution using a finite difference method and showed a relationship between the accuracy of predicted values and the level of the uncertainties included and also between minimum predictive interval and the level of the uncertainties. Shoji [5] finally pointed out that Laplace transformation is promising in treating one dimensional IHTP.

Imber [6] employed an approximate function expressed by a polynomial function to estimate the known temperature change with time in place of the discrete values which were used by Buggraf [3], Sparrow et al. [4] and Shoji [5], and applied it to the subsidiary equation to get the inverse solution explicitly. In his method, a relationship between temperatures measured at two different points is assumed to avoid divergence of the inverse solution. His solution does not need any iterative calculation.

Imber [7], furthermore, extended his procedure, using Laplace transformation, for one-dimensional IHTP to one for two- and three-dimensional IHTPs, since it is relatively simple to extend two- and three-dimensional IHTPs for the cases where the geometrical configurations involved are not complicated.

As for numerical methods, Heieh and Su [8], Bell [9] and Lithouhi and Beck [10] solved two-dimensional IHTP using finite-difference method, while Shoji and Ono [11] used on boundary element method. Frankel et al. [12] first approximated the temperature change at a point using polynominal series of Cheviseff, and then gave the solution of IHTP by determining each coefficient of polynominal series, in order to minimize a weighted residual in the governing equation. Chen and Chang [13] developed a little different method for one-dimensional IHTP by combining Laplace transformation and the Galerkin weighted residual process. This method needs several measuring points in a solid at which the known values can be provided although the number of measuring points is two for a finite body and one for the infinite body. Chen and Chang [13] showed good agreement between estimates and the exact solution for non-dimensional times of $\tau = 1$ and 5, through it is not mentioned that what kind of uncertainties are merged into the temperatures used. From the viewpoint that the inverse solution for short time may be generally needed during a transient heat conduction, non-dimensional times of $\tau = 1$ and 5 seem to be too large, because heat transfer process may reach close to steady state, corresponding to a gradual change in the temperature with time.

The present study also has started with the same subsidiary solution as governing equation for one-dimensional heat conduction as explained by Sparrow et al. [4], Shoji [5], Imber [6] and Chen and Chang [13]. The main difference from the former researches is to employ equation expressed by a half polynomial series of time to approximate the known values. As a result, the inverse solution can be obtained explicitly, so that no iterative calculation is required and the calculation of the solution becomes very quick. The minimum predictive time and the stability of the solution and the uncertainty in predicted values will be discussed using the known temperatures including uncertainties.

2. Analysis of one-dimensional heat conduction

One-dimensional heat conduction equation with constant properties can be written in a non-dimensional form as:

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\xi^2} \tag{1}$$

A subsidiary form after Laplace transformation can be expressed for an initial condition of $\theta = 0$, which does not lose any generality for a constant initial temperature, becomes

$$\frac{d^2\bar{\theta}}{d\xi^2} - p^2\bar{\theta} = 0 \tag{2}$$

The general solution of Eq. (2) can be easily given as:

$$\bar{\theta}(\xi, s) = A\, e^{p\xi} + B\, e^{-p\xi} \tag{3}$$

where $p^2 = s$, and s is Laplace's operator and A and B are integral constants subject to surface conditions.

2.1. Solution for finite plate

In the case of IHTP for a finite plate, two known temperatures in the plate are necessary at least to close these equations. Therefore, let the two temperatures at two different points be:

$$\theta(\xi_n, \tau) = f_n(\tau) \quad \text{at } \xi = \xi_n, n = 1, 2 \tag{4a}$$

and then its subsidiary form becomes:

$$\theta(\xi_n, s) = \bar{f}_n(s) \quad \text{at } \xi = \xi_n, \quad n = 1, 2 \tag{4b}$$

where, $0 < \xi_1 < \xi_2$.

By substituting Eq. (4b) in Eq. (3), the two integral constants A and B in Eq. (3) can be determined and then a final solution for the temperaturet any point becomes:

$$\bar{\theta}(\xi, s) = \frac{\bar{f}_1(s)\sinh\{p(\xi_2 - \xi)\} + \bar{f}_2(s)\sinh\{p(\xi - \xi_1)\}}{\sinh\{p(\xi_2 - \xi_1)\}} \tag{5}$$

and the solution for the heat flux $\Phi(= q/(\lambda T_0/L) = -\partial\theta/\partial\xi)$ is also given as:

$$\bar{\Phi}(\xi, s) = p\frac{\bar{f}_1(s)\cosh\{p(\xi_2 - \xi)\} - \bar{f}_2(s)\cosh\{p(\xi - \xi_1)\}}{\sinh\{p(\xi_2 - \xi_1)\}} \tag{6}$$

The objective in the IHTP, in general, is to predict the unknown values in the region not included in, $\xi_1 \le \xi \le \xi_2$, especially to determine the surface conditions such as surface temperature and surface heat flux. Therefore, let $\xi = 0$ in Eqs. (5) and (6), we can express the surface temperature and surface heat flux in a subsidiary form as:

$$\bar{\theta}_w(s) = \frac{\bar{f}_1(s)\sinh(p\xi_2) - \bar{f}_2(s)\sinh(p\xi_1)}{\sinh\{p(\xi_2 - \xi_1)\}} \tag{7}$$

$$\bar{\Phi}_w(s) = p\frac{\bar{f}_1(s)\cosh(p\xi_2) - \bar{f}_2(s)\cosh(p\xi_1)}{\sinh\{p(\xi_2 - \xi_1)\}} \tag{8}$$

The actual surface conditions can be obtained by executing inverse Laplace transformation defined by the following integral.

$$F(\xi, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\tau}\bar{F}(\xi, s)\, ds \tag{9}$$

2.2. Approximate equation for temperatures at a measuring point

In order to execute Eq. (9), we first have to give two known functions, $\bar{f}_n(s)$, $n = 1, 2$, included in $\bar{\theta}_{\bar{w}}(s)$ and $\bar{\Phi}_{\bar{w}}(s)$ of Eqs. (7) and (8) explicitly, which are previously determined from a temperature change measured at a point. Therefore, we can approximate the temperature change at the point with a half polynomial series of time given as:

$$f_n^{(1)}(\tau) = \sum_{k=0}^{N} a_{k,n}^{(1)}\tau^{\frac{k}{2}}, \quad n = 1, 2 \tag{10a}$$

$$f_n^{(2)}(\tau) = \sum_{k=1}^{N} a_{k,n}^{(2)}\tau^{\frac{k}{2}}, \quad n = 1, 2 \tag{10b}$$

$$f_n^{(3)}(\tau) = \sum_{k=0}^{N} a_{k,n}^{(3)}\left(\tau - \tau_n^*\right)^{\frac{k}{2}}, \quad n = 1, 2 \tag{10c}$$

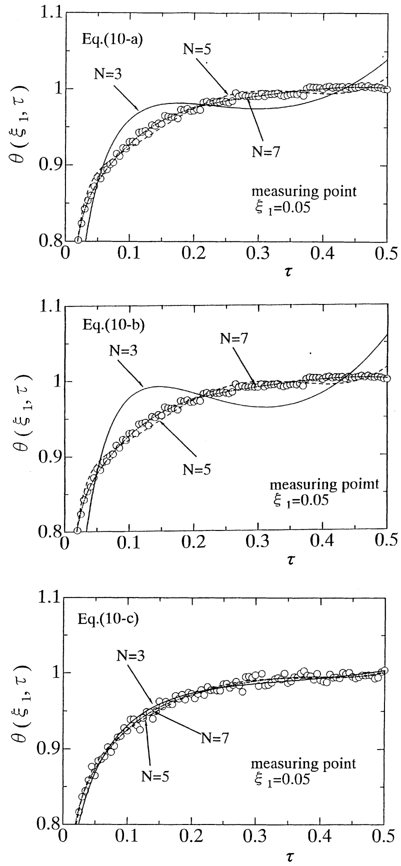

Fig. 1. Approximate function for temperature change measured at a point.

where coefficients $a_{k,n}^{(i)}$ can be determined using least mean square method from the measured temperature and N gives the degree of the polynomial series.

Three different forms as shown in Eqs. (10a)–(10c), incidentally may be possible as an approximate function; the first one does not require any constraint, the second one satisfies an initial condition of $f_n(0) = 0$ and the third one takes into account a time lag, τ_n^*, which it takes for a temperature to be monitored at the measuring point. This time lag can be determined as $\mathrm{erfc}(\xi_n/2\sqrt{\tau_n^*}) = \min(\theta)$.

The differences in the three equations may be worth mentioning. We notice that the three equations are identical in a basic form, except for $a_{0,n}^{(2)} \equiv 0$ in Eq. (10b) and $\tau_n^* \neq 0$ in Eq. (10c). Therefore, we may express Eq. (10c) only after this and omit superscript in the coefficients to avoid a complexity. In addition to this, new coefficients, $b_{k,i} = a_{k,i}\Gamma(\frac{k}{2}+1),\ i = 1, 2$, are introduced for convenience by using Gamma function. Consequently, the subsidiary form of Eq. (10c) becomes as:

$$\bar{f}_n(s) = e^{s\tau_n^*}\sum_{k=0}^{N} b_{k,n}/s^{\left(\frac{k}{2}+1\right)}, \quad n = 1, 2 \tag{11}$$

It would be necessary to say why the half polynomial series for time as shown in Eq. (10) is adopted instead of a polynomial series for time as a functional form approximating the temperature change at the point. The reason is that a general solution for one-dimensional heat conduction is provided in a functional form of $T = f(x/\sqrt{at})$.

2.3. Procedure to determine coefficients

The coefficients of $a_{k,n}$ are greatly subject to the significant digits of the data or precision of measured values. We can consider two different cases: one is cut-off of values calculated from the exact solution at a certain significant digit, namely $\theta(\xi_n,\ \tau) = \mathrm{Int}(\theta_{\mathrm{exact}}(\xi_n,\ \tau) \times 10^{N_{\mathrm{sf}}})/10^{N_{\mathrm{sf}}}$ (N_{sf} means the level of the significant digits) and the other is to superpose with a certain disturbance on the exact solution, namely $\theta(\xi_n,\ \tau) = \theta_{\mathrm{exact}}(\xi_n,\ \tau) + 0.005\varepsilon(m = 0,\ \sigma = 1)$ (m and σ are average and standard deviation for a value of ε) on the exact value, since the level of significant digits of two or three would be enough for the case of a temperature measured by a thermocouple. $\theta(\xi_n,\ \tau) = \theta_{\mathrm{exact}}(\xi_n,\ \tau) + 0.005\varepsilon(m = 0,\ \sigma = 1)$ just corresponds to two significant digits.

Fig. 1 shows a change in temperature measured at a point with time and the corresponding curves given by Eqs. (10a)–(10c) with different orders of N under the condition that the surface temperatures at both ends are suddenly raised from $\theta = 0$ to $\theta(0,\tau) = \theta(1,\tau) = 1$.

Fig. 1 shows that Eqs. (10a)–(10c) are improved with an increase in N, but the improvement would be saturated around $N = 5$–7 beyond which the improvement is not accepted greatly. Therefore, the order of approximate equation can be considered to be $N = 5$–7. The accurate level of approximation becomes the same for each equation with $N = 5$–7. There is little difference among Eqs. (10a)–(10c) in approximating the temperature measured when employing $N = 5$–7.

Comparing the accuracy of the values predicted by using either cut-off data or disturbed data with the same significant values, one notices that the disturbed data give worse approximation though figure to show it is omitted here. Since in general, the worse approximation would result in a wor inverse solution, accuracy of the inverse solution will be discussed using the worse approximation obtained from the disturbed data, namely $\theta(\xi_n,\ \tau) = \theta_{\mathrm{exact}}(\xi_n,\ \tau) + 0.005\varepsilon(m = 0,\ \sigma = 1)$.

2.4. Inverse Laplace transformation and characteristics of the solution

Substituting Eq. (11) into Eqs. (7) and (8), and then into Eq. (9), we get

$$\theta_{\mathrm{w}}(\tau)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathrm{e}^{s\tau}\frac{\mathrm{e}^{s\tau_2^*}\sum_{k=0}^{N}b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)}\sinh(p\xi_2)}{\sinh\{p(\xi_2-\xi_1)\}}\,\mathrm{d}s-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathrm{e}^{s\tau}\frac{\mathrm{e}^{s\tau_1^*}\sum_{k=0}^{N}b_{k,\,2}/s^{\left(\frac{k}{2}+1\right)}\sinh(p\xi_1)}{\sinh\{p(\xi_2-\xi_1)\}}\,\mathrm{d}s \tag{12}$$

$$\Phi_{\mathrm{w}}(\tau)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathrm{e}^{s\tau}p\frac{\mathrm{e}^{s\tau_2^*}\sum_{k=0}^{N}b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)}\cosh(p\xi_2)}{\sinh\{p(\xi_2-\xi_1)\}}\,\mathrm{d}s-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathrm{e}^{s\tau}p\frac{\mathrm{e}^{s\tau_1^*}\sum_{k=0}^{N}b_{k,\,2}/s^{\left(\frac{k}{2}+1\right)}\cosh(p\xi_1)}{\sinh\{p(\xi_2-\xi_1)\}}\,\mathrm{d}s \tag{13}$$

Since the integration of Eqs. (12) and (13) can be given by the sum of the residues of the integrands at its poles within the contour shown in Eqs. (12) and (13), one has to look for the poles and then calculate residues to obtain the exact inverse solution. We would face some difficulty in calculating the residues; if the exponet of s becomes even number, then the poles are easily found to give the residues, while if the exponent becomes odd number, then there is a branch at $s=0$, consequently the solution can not be expressed by any elementary function but by a series function [15], only.

As for the characteristics of the solutions, incidentally, these are found to be split into two parts; one is a group composed of the residue only at the pole of $s=0$, and the other consists of the residues at $s\neq 0$. The basic functional form for the residues at $s\neq 0$ can be expressed as $\exp(-n^2\tau)/n^k$ $(n,k\geq 1)$, which are quickly decreasing with increasing τ, which in terns corresponds to decreasing s, and higher order of n and k. Inversely, these values hardly converge with decreasing τ, namely large values of s and are strongly subject to uncertainties included in the value measured. The fact that the integration of Eqs. (12) and (13) does not converge for a large value of s, is mathematically verified ([2], Chapter 1). In other words, there is a limiting time beyond which these integrals can be converged to give the surface temperature and surface heat flux.

Therefore, we choose an easier way rather executing the exact one directly, that is, we first expand the integrants in Eqs. (12) and (13) around $s=0$ since the measured temperature is expressed by a half polynomial function of s, and then conduct the integration. As a result, we can obtain the following solutions for the surface temperature and heat flux, explicitly:

$$\theta_{\mathrm{w}}(\tau)=\sum_{j=-1}^{N}C_{j,\,21}\left(\tau-\tau_2^*\right)^{\frac{j}{2}}/\Gamma\left(\frac{j}{2}+1\right)-\sum_{j=-1}^{N}C_{j,\,12}\left(\tau-\tau_1^*\right)^{\frac{j}{2}}/\Gamma\left(\frac{j}{2}+1\right) \tag{14}$$

$$\Phi_{\mathrm{w}}(\tau)=\sum_{j=-1}^{N}D_{j,\,21}\left(\tau-\tau_2^*\right)^{\frac{j}{2}}/\Gamma\left(\frac{j}{2}+1\right)-\sum_{j=-1}^{N}D_{j,\,12}\left(\tau-\tau_1^*\right)^{\frac{j}{2}}/\Gamma\left(\frac{j}{2}+1\right) \tag{15}$$

Each coefficient in Eqs. (14) and (15) and the derivation are summarized in the Appendix A.

2.5. Solution for semi-infinite body

A general solution for semi-infinite body can be easily obtained by letting $A=0$ in Eq. (3). The temperature change needed becomes only one measure at a point. Provided that the temperature change at $\xi=\xi_1$ can be approximated by Eq. (10) with a time lag, the solutions for the surface temperature and the surface heat flux become, respectively as:

$$\theta_{\mathrm{w}}(\tau)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathrm{e}^{s\left(\tau-\tau_1^*\right)}\mathrm{e}^{\,p\xi_1}\sum_{k=0}^{N}b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)}\,\mathrm{d}s \tag{16}$$

$$\Phi_{\mathrm{w}}(\tau)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathrm{e}^{s\left(\tau-\tau_1^*\right)}p\,\mathrm{e}^{\,p\xi_1}\sum_{k=0}^{N}b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)}\,\mathrm{d}s \tag{17}$$

The same procedure as the finite plate gives the inverse solutions for semi-infinite body as:

$$\theta_{\mathrm{w}}(\tau)=\sum_{j=-1}^{N}E_j\left(\tau-\tau_1^*\right)^{\frac{j}{2}}/\Gamma\left(\frac{j}{2}+1\right) \tag{18}$$

$$\Phi_{\mathrm{w}}(\tau)=\sum_{j=-1}^{N}G_j\left(\tau-\tau_1^*\right)^{\frac{j}{2}}/\Gamma\left(\frac{j}{2}+1\right) \tag{19}$$

Table 1
Boundary condition and exact solution (initial condition $\theta = 0$ for all cases)

	Boundary condition ($0 < \tau$)	Parameters
	Finite body ($0 \le \xi \le 1$)	
Case 1	$\theta = 1$; $\xi = 0, 1$	$\theta = T/T_0$ $\Phi = qL/\lambda T_0$
Case 2	$\theta = \tau$; $\xi = 0, 1$	$\theta = aT_0/kL2$ $\Phi = aq/\lambda k T_0 L$
Case 3	$\theta = \tau$; $0 < \tau < 1$, $\xi = 0, 1$ $\theta = 2 - \tau$; $1 < \tau < 2$	$\theta = aT/kT_0L^2$ $\Phi = aq/\lambda k T_0 L$
	Semi-infinite body ($0 \le \xi$)	
Case 4	$\theta = 1$; $\xi = 0$	$\theta = T/T_0$ $\Phi = qL/\lambda T_0$
Case 5	$\theta = \tau$; $\xi = 0$	$\theta = aT/kT_0L^2$ $\Phi = aq/\lambda k T_0 L$
Case 6	$\theta = \tau$; $0 < \tau < 1$, $\xi = 0$ $\theta = 2 - \tau$; $1 < \tau < 2$	$\theta = aT/kT_0L^2$ $\Phi = aq/\lambda k T_0 L$

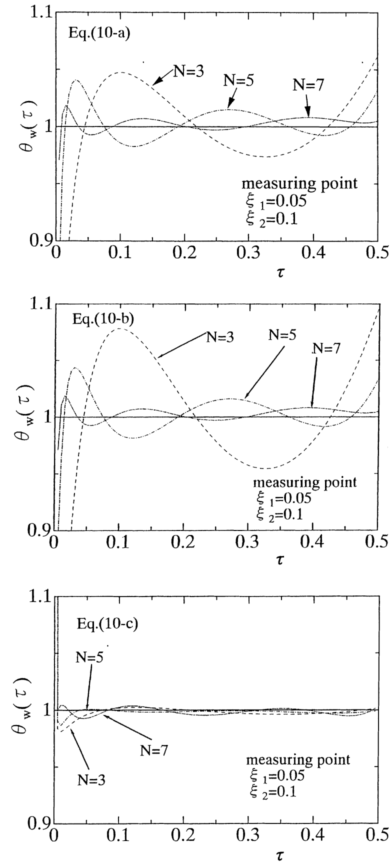

Fig. 2. Estimated surface temperature.

3. Inverse solution and representative problems

3.1. Method for calculation

The way to solve representative problems becomes:

1. Determine each coefficient of $a_{i,n}$ (or $b_{i,n}$) in Eq. (10), calculated from the temperature measured at the point ξ_1 or ξ_2, for example, by the least mean square method.
2. Expand integrants in Eqs. (12) and (13) around $s = 0$ in a series in which the coefficients are summarized in the Appendix A.
3. Follow the integration given by Eqs. (12) and (13) after some multiplication of coefficients, resulting into the final forms for the surface temperature and heat flux given by either Eqs. (14) and (15) for the finite body or by Eqs. (18) and (19) for the semi-infinite body.

3.2. Representative problems

Inverse solutions will be treated for a combination of an initial temperature of $\theta = 0$ and the first kind of boundary listed in Table 1, whose solutions are given in [14] or can be derived from it.

3.3. Inverse solution calculated for case 1

Fig. 2 shows a comparison between the exact value for case 1 and the corresponding values estimated by using three different types of approximate Eqs. (10a)–(10c). The value of N in Fig. 2 gives the order of approximate equation and the values of ξ_1 and ξ_2 correspond to the position of the measuring point. The values of $\theta(\xi_n, \tau) = \theta_{\text{exact}}(\xi_n, \tau) + 0.005\varepsilon$ are employed as the values measured.

Fig. 2 shows that the estimated values are improved, approaching to the exact solution with an increase in the value of N. In addition, the values estimated by Eq. (10c), from Fig. 2, are found to be in good agreement with the exact one compared with the values from Eqs. (10a) and (10b). It may be worth mentioning that the improvement of solution is not expected beyond $N = 7$ and a suitable order of approximate equation seems in the range 5–7.

Table 2
Minimum predictive time of inverse solution

	Finite body					Semi-infinite body		
	Case 1			Case 2	Case 3	Case 4	Case 5	Case 6
N	Eq. (10a)	Eq. (10b)	Eq. (10c)			Eq. (10c)		
3	0.0410	0.0421	0.0051	0.0057	0.2156	0.0104	0.0300	0.0365
4	0.0212	0.0213	0.0051	0.0057	0.2070	0.0104	0.0296	0.0368
5	0.0132	0.0132	0.0052	0.0056	0.0783	0.0104	0.0294	0.0208
6	0.0090	0.0090	0.0052	0.0047	0.0162	0.0105	0.0256	0.0208
7	0.0074	0.0074	0.0052	0.0076	0.0319	0.0100	0.0050	0.0208
8	0.0053	0.0053	0.0052	0.0057	0.0155	0.0103	0.0371	0.0208
9	0.0050	0.0050	0.0013	0.0203	0.0054	0.0103	0.0390	0.0208

4. Evaluation of estimated value

4.1. Minimum predictive time

It is mathematically proved [2] that no inverse solution exists at $\tau = 0$ and the solution can converge beyond a limiting time. Therefore, a minimum predictive time is an important factor in evaluating the inverse solution. One may adopt the minimum predictive time, τ_1 at which Eq. (14) is first satisfied within a relative difference of 0.01 between the exact and the estimated values and from which Eqs. (14) and (15) are validated.

Table 2 shows the minimum predictive time for Eqs. (10a)–(10c) against the value of N. In Table 2, the minimum predictive times for other cases are also listed that will be discussed later.

Table 2 shows that the best minimum time can be obtained by using Eq. (10c) and this time is independent of N, while for the other two equations, Eqs. (10a) and (10b), the minimum predictive time is improved with an increase in the value of N and finally reaches the same level as for Eq. (10c).

4.2. Standard deviation of inverse solution

In order to evaluate the inverse solution, we may introduce standard deviation as:

$$\sigma = \sqrt{\frac{1}{(\tau_2 - \tau_1)} \int_{\tau_1}^{\tau_2} \left(\theta_{\mathrm{w,\,exact}}(\tau) - \theta_{\mathrm{w,\,cal}}(\tau)\right)^2 \mathrm{d}\tau}$$

where τ_2 is defined as the time when the measurement is ended for the semi-infinite body, while for the finite body, 90% of the full time τ_2 is employed to avoid the effect of the ending time on the estimated temperature.

Fig. 3 shows the standard deviation between the exact solution of $\theta_{\mathrm{w,\,exact}} = 1$ and the values estimated for the case 1 plotted against the order of N. It is found from Fig. 3 that the inverse solution using Eq. (10c) predicts the exact values at a good precision at a small number of N and the minimum deviation of $\sigma = 0.003$ appears at $N = 6$ beyond which the accuracy of prediction is inversely not expected to improve any more. Taking into account the fact that the deviation in prediction reaches the same level as deviation of approximate equation of Eq. (10), we can not expect more improvement in this method.

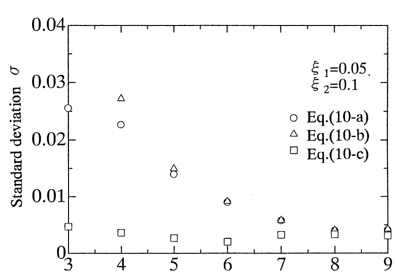

Fig. 3. Standard deviation for each equation.

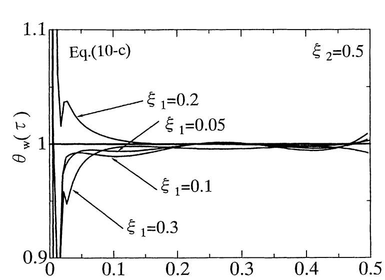

Fig. 4. Effect of position of measurement.

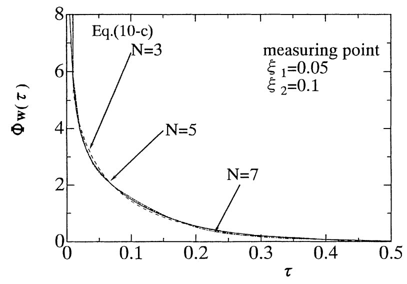

Fig. 5. Estimation of heat flux in case 1.

4.3. Recommended approximate equation

There are three different equations as given by Eqs. (10a)–(10c) in approximating the temperature change at the measuring point. Taking into account both standard deviation and the minimum predictive time, one can recommend Eq. (10c) among three equations and then the values of $N = 5$ or 6 may be thought to be enough.

4.4. Effect of position of measuring temperature

Fig. 4 shows the effect of position of temperature measurement on the inverse solution when Eq. (10c) with $N = 5$ is used. The position of $\xi_2 = 0.5$ in Fig. 4 corresponds to the farthest point from the surface in a finite body.

Fig. 4 shows that the inverse solutions for $\xi_1 = 0.05$ and 0.1 are in good agreement with the exact value while the minimum predictive time is slightly deteriorated, but for $\xi_1 \geqq 0.2$, both the solutions and the time are not recommended because of a large deviation and a delayed time. Consequently, for any combination of $\xi_1 \leqq 0.1$ and any value of ξ_2, the inverse solution obtained may become available.

4.5. Prediction of surface heat flux

Fig. 5 shows a comparison of the surface heat flux calculated from Eq. (15) using Eq. (10c) recommended here and the exact value in the case 1. It is found from Fig. 5 that the heat flux can be well predicted when using $N = 5$ and 7.

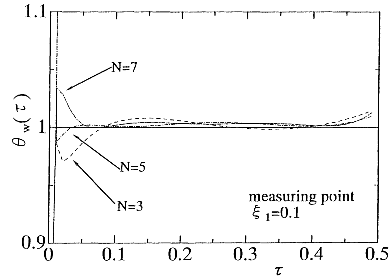

Fig. 7. Comparison of solution for case 4.

5. Calculation for other cases

The method, using Eq. (10c), is applied to five other cases listed in Table 1 in order to check its applicability. The results for each case are shown in Figs. 6–8 (The results for cases 3 and 5 are omitted here, because it is almost the same as that for cases 2 and 6.) In addition to this, the standard deviations for each case are also calculated as listed in Table 3.

Figs. 6–8 and Table 3 show that the present method can be considered to be applicable to the other cases except for the cases 3 and 6. The reason why the precision level for the cases 3 and 6 is inferior to that for the other cases, is that of its temperature change, that is the first derivative of the temperature becomes discontinuous at $\tau = 1$. In other words, its sharp discontinuity makes a sharp change in the surface temperature decay at a measuring point. As the result, the estimated values largely deviate from the exact value at this point, as shown in Fig. 8. From the engineering point of view, this method may be promising and powerful as one of the IHTP solutions, since most of the cases are actually subject to a continuous temperature change.

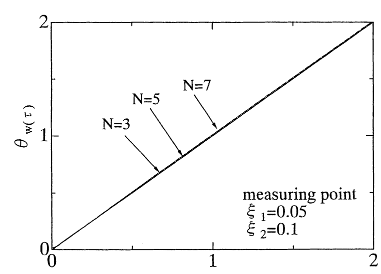

Fig. 6. Comparison of solution for case 2.

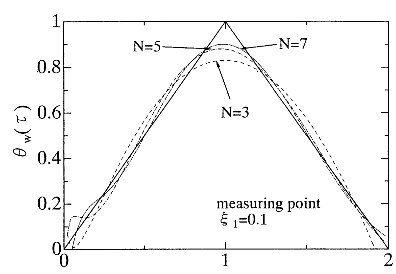

Fig. 8. Comparison of solution for case 6.

Table 3
Standard deviation for each example

	Finite body			Semi-infinite body		
N	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
3	0.0047	0.0091	0.0679	0.0085	0.0004	0.0655
4	0.0036	0.0093	0.0688	0.0052	0.0004	0.0652
5	0.0027	0.0093	0.0436	0.0024	0.0005	0.0405
6	0.0020	0.0093	0.0330	0.0026	0.0005	0.0335
7	0.0032	0.0094	0.0314	0.0072	0.0006	0.0307
8	0.0033	0.0094	0.0220	0.0197	0.0008	0.0221
9	0.0031	0.0095	0.0209	0.0169	0.0009	0.0213

6. Comparison of existing method using Laplace transformation

Shoji [5] expanded functions of sinh and cosh in Eqs. (7) and (8) around $s = 0$ in series and then reformed a sum of $s^n \bar{f}_n(s)$, $n = 1, 2$, which means nth order derivative of $f_n(\tau)$ with respect to time. Therefore, the Shoji method gave the inverse solution as a term of numerical derivative of the temperature change measured at a point. The Shoji and the present methods can be thought as basically identical except that the numerical derivative was employed in place of integration as shown in Eqs. (12) and (13).

Comparison of the present and the Shoji procedures shows that (1) accuracy of the inverse solution predicted by using the temperature with an accuracy of 2 or 3 significant digits can be improved by the present one, (2) there is little difference in the minimum predictive time, (3) the present method shows higher robust for the deterioration of the accuracy of the measured temperature since uncertainties included in numerical derivative propagates with a successive iteration of time.

Imber [6] also obtained the inverse solution using the same procedure as the present one, except that a polynomial function with time was adopted in place of a function of a half polynomial series of time, as shown in Eq. (10), to approximate the measured temperature change and no consideration was given for a time lag. Therefore, his method resulted in worse inverse solution than the present one. This fact may mean that a lower order of half polynomial series plays an essential role in the region of a short time and correctly approximates a functional form of solution, $\theta = f(\xi/\sqrt{\tau})$.

7. Conclusions

1. The inverse solution for one-dimensional heat conduction is explicitly given using Laplace transformation and a suitable functional form expressed by half polynomial series of time to approximate change of the measured temperature with time.
2. The approximate equation which takes into account time lag gives the best inverse solution.
3. The explicit solution requires no iterative process and then gives the surface condition of the temperature as well as heat flux quickly.
4. The inverse solution obtained is robust against a disturbance included in the temperature change.

Acknowledgements

The author would like to acknowledge Professor of Saga University, Department of Mathematics, Y. Furusho who gave his valuable advice on Laplace transformation, and then Research associate of Saga University, Mr. Y. Mitsutake, and Dr. Arima and Graduate student, Mr. T. Nishimoto for their valuable support in executing calculation.

Appendix A

A.1. Expansion of integrant in a series

Expansion of integrants in Eqs. (12) and (13) around $s = 0$ in a series

$$\sinh(p\xi_n)/\sinh\{p(\xi_2 - \xi_1)\} = \sum_{i=0}^{\infty} c_{i,n} s^i, \quad n = 1, 2 \quad \text{(A1)}$$

$$\cosh(p\xi_n)/\sinh\{p(\xi_2 - \xi_1)\} = \sum_{i=0}^{\infty} d_{i,n} s^i, \quad n = 1, 2 \quad \text{(A2)}$$

where, $n = 1, 2$ give the position of the measurement at $\xi = \xi_1$ and ξ_2.

The coefficients of $c_{i,n}$ and $d_{i,n}$ in Eqs. (A1) and (A2) can be given as:

$$c_{0,n} = \xi_n/(\xi_2 - \xi_1)$$

$$c_{i,n} = c_{0,n}(\xi_2 - \xi_1)^{2i}\left(\frac{1}{(2i+1)!}c_{0,n}^{2i} - \sum_{j=0}^{i-1} A_{2(i-j)} \frac{c_{0,n}^{2j}}{(2j+1)!}\right), \quad i \geq 1, n = 1, 2$$

$$d_0 = 1/(\xi_2 - \xi_1)$$

$$d_{i,\,n} = (\xi_2 - \xi_1)^{2i-1}\left(\frac{1}{(2i)!}c_{0,\,n}^{2i} - \sum_{j=0}^{i-1} A_{2(i-j+1)}\frac{c_{0,\,n}^{2j}}{(2j)!}\right),$$

$$i \geq 1, n = 1, 2$$

where

$$A_2 = \frac{1}{3!},\ A_4 = \frac{1}{5!} - \left(\frac{1}{3!}\right)^2,\ A_6 = \frac{1}{7!} - 2\frac{1}{3!}\frac{1}{5!}$$

$$A_8 = \frac{1}{9!} - 2\frac{1}{3!}\frac{1}{7!} - \left(\frac{1}{5!}\right)^2 + 3\frac{1}{5!(3!)^2} - \left(\frac{1}{3!}\right)^4,$$

$$A_{10} = \frac{1}{11!} - 2\left(\frac{1}{3!}\frac{1}{9!} + \frac{1}{5!}\frac{1}{7!}\right) + 3\left(\frac{1}{3!(5!)^2} + \frac{1}{(3!)^3 5!}\right) - 4\frac{1}{5!(3!)^3} + \left(\frac{1}{3!}\right)^5.$$

The same procedure for Eq. (16) gives expansion as:

$$\mathrm{e}^{\,p\xi_1} = \sum_{n=0}^{\infty} \mathrm{e}_n p^n, \quad \mathrm{e}_n = \frac{\xi_1^n}{n!} \tag{A3}$$

A.2. Multiplication of coefficients

The integrants in Eqs. (12) and (13) can be reformed by using the coefficients of $b_{i,\,j}$, $c_{l,\,j}$, and $d_{l,\,j}$ as:

$$\sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} \sinh(p\xi_2)/\sinh\{p(\xi_2 - \xi_1)\}$$

$$= \sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} \sum_{i=0}^{\infty} c_{i,\,2}s^i = \sum_{j=-1}^{N} C_{j,\,12}s^{\left(\frac{j}{2}+1\right)}$$

$$C_{-1,\,12} = \sum_{k=0}^{Nk} b_{2k+1,\,1}c_{k+1,\,2}, \quad j = -1,$$

$$Nk = \mathrm{Int}\{(N-1)/2\}$$

$$C_{j,\,12} = \sum_{k=0}^{Nk} b_{2k+j,\,1}c_{k,\,2}, \quad j \geq 0, Nk = \mathrm{Int}\{(N-j)/2\}$$

As for $C_{-1,21}$, $C_{j,21}$ at other point, the values are subject to the subscripts.

$$\sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} p\cosh(p\xi_2)/\sinh\{p(\xi_2 - \xi_1)\}$$

$$= \sum_{j=-1}^{N} D_{j,\,12}s^{\left(\frac{j}{2}+1\right)}$$

$$D_{-1,\,12} = \sum_{k=0}^{Nk} b_{2k+1,\,1}d_{k+1,\,2}, \quad j = -1,$$

$$Nk = \mathrm{Int}\{(N-1)/2\}$$

$$D_{j,\,12} = \sum_{k=0}^{Nk} b_{2k+j,\,1}d_{k,\,2}, \quad j \geq 0, Nk = \mathrm{Int}\{(N-j)/2\}$$

As for $D_{-1,\,21}$, $D_{j,\,21}$, the values are subject to the subscripts.

The coefficients of E_j and G_j in Eqs. (18) and (19) are also given as:

$$\sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} \mathrm{e}^{\,p\xi_1} = \sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} \sum_{i=0}^{\infty} \mathrm{e}_i s^{i/2}$$

$$= \sum_{j=-1}^{Nj} E_j s^{\left(\frac{j}{2}+1\right)}$$

$$E_{-1} = \sum_{k=0}^{Nk} b_{k,\,1}\mathrm{e}_{k+1}, \quad j = -1, Nk = N$$

$$E_j = \sum_{k=0}^{Nk} b_{k+j,\,1}\mathrm{e}_k, \quad j \geq 0, Nk = N - j$$

$$\sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} p\,\mathrm{e}^{\,p\xi_1} = p\sum_{k=0}^{N} b_{k,\,1}/s^{\left(\frac{k}{2}+1\right)} \sum_{i=0}^{\infty} \mathrm{e}_i s^{i/2}$$

$$= \sum_{j=-1}^{Nj} G_j s^{\left(\frac{j}{2}+1\right)}$$

$$G_{-1} = \sum_{k=0}^{Nk} b_{k,\,1}\mathrm{e}_k, \quad j = -1, Nk = N$$

$$G_j = \sum_{k=0}^{Nk} b_{k+j+1,\,1}\mathrm{e}_k, \quad j \geq 0, Nk = N - j - 1$$

References

[1] J.V. Beck, Inverse Heat Conduction, Wiley/Interscience, NY, 1985.

[2] O.M. Alifanov, Inverse Heat Transfer Problems, Springer-Verlag, Berlin, 1994, pp. 5–8.

[3] O.R. Burggraf, An exact solution of the inverse problem in heat conduction theory and application, J. of Heat Transfer 86 (1964) 373–382.

[4] E.M. Sparrow, A. Haji-Sheikh, T.S. Lundgren, The inverse problem in transient heat conduction, J. of Heat Transfer 86 (1964) 369–375.

[5] M. Shoji, Study of inverse problem of heat conduction (in Japanese), Trans. JSME 44 (381) (1978) 1633–1643.

[6] M. Imber, J. Khan, Prediction of transient temperature distributions with embedded thermo-couples, AIAA Journal 10 (6) (1972) 784–789.

[7] M. Imber, Temperature extrapolation mechanism for two-dimensional heat flow, AIAA Journal 12 (8) (1974) 1089–1093.

[8] C.K. Hsieh, K.C. Su, A methodology of predicting cavity geometry based on scanned surface temperature data prescribed surface temperature at the cavity side, J. Heat Transfer 102 (2) (1980) 324–329.

[9] G.E. Bell, An inverse solution for the steady temperature field within a solidified layer, Int. J. Heat Mass Transfer 27 (12) (1984) 2331–2337.

[10] B. Lithouhi, J.V. Beck, Multinode unsteady surface element method with application to contact conductance problem, J. Heat Transfer 108 (2) (1986) 257–263.

[11] M. Shoji, N. Ono, Application of the boundary element to the inverse problem of heat conduction (in Japanese), Trans. JSME, Ser. B 54 (506) (1988) 2893–2900.

[12] J.I. Frankel, A global time treatment for inverse heat conduction problems, J. of Heat Transfer 119 (4) (1997) 673–683.

[13] H.K. Chen, S.M. Chang, Application of the hybrid method to inverse heat conduction preblems, Int. J. Heat Mass Transfer 33 (4) (1990) 621–628.

[14] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solid, Oxford University Press, London, 1959.

[15] S. Moriguti, et al., Mathematical Formula, vol. 1, Iwanami Zennsho, Tokyo, 1956 (in Japanese).